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Coherent seeding of self-modulated plasma wakefield accelerators^{a)}

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The growth of the beam self-modulation and hosing instabilities initiated by a seed wakefield is examined. Although the growth rates for the self-modulation and hosing instabilities are comparable, it is shown that an externally excited wakefield can be effective in selectively seeding the beam radial self-modulation, enabling the beam to fully modulate before strong beam hosing develops. Methods for coherent seeding are discussed.

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I. INTRODUCTION

Plasma-based acceleration is realized by using an intense laser¹ or charged-particle beam²⁻⁴ to excite large amplitude electron plasma waves with relativistic phase velocity. The electric field amplitude of the plasma wave can be several orders of magnitude greater than conventional accelerators, on the order of $E_0 = cm_e\omega_p/e$, or $E_0[\text{V/m}] \simeq 96\sqrt{n_0[\text{cm}^{-3}]}$, where $\omega_p = (4\pi n_0 e^2/m_e)^{1/2}$ is the plasma frequency, n_0 is the ambient electron number density, m_e and e are the electron rest mass and charge, respectively, and c is the speed of light in vacuum. Owing to the ultra-high field gradients, plasma-based accelerators are considered a candidate technology for the next generation lepton colliders to expand the energy frontier of high energy physics experiments.⁵⁻⁷

The energy gain in a single-stage plasma-accelerator is limited by the drive beam energy. Proton accelerators, such as those available at CERN (European Organization for Nuclear Research), are able to generate multi-TeV beams with tens of kJs of energy, and it has been proposed to use highly relativistic proton beams to drive a plasma accelerator.^{8,9} The plasma provides a mechanism to transfer energy from the proton beam to a lepton beam over a relatively short interaction distance. Efficient plasma wave excitation requires beam drivers with spatial structure on the scale of the plasma skin depth, and compact, high-gradient plasma accelerators (i.e., operating at high plasma density) require short drive beams. Generating short proton beams (or proton beams with spatial structure at the plasma frequency) is challenging, for interesting plasma densities, and it has been proposed to rely on a beam-plasma instability to modulate the beam at the plasma wavelength $\lambda_p = 2\pi c/\omega_p$, driving a large amplitude wave.¹⁰⁻¹² Proof-of-principle experiments to study the physics of beam self-modulation using lepton beams have also been proposed.¹³

The growth rate of the self-modulation instability was calculated in Ref. 14. It was also shown that the phase velocity of the plasma wave excited by a beam undergoing self-modulation is significantly less than the velocity of the drive beam, limiting the energy gain.^{14,15} Tapered plasmas may be considered to control the phase velocity of the self-modulated beam-driven plasma wave, although sufficiently large background plasma density variations will suppress the self-modulation.¹⁶ To avoid a witness bunch interacting with the slow-phase-velocity plasma wave driven by the self-modulating beam, one may consider injection of a witness bunch after saturation of the instability, when the drive beam is no

longer strongly evolving and the phase velocity approaches the drive beam velocity.

Transverse stability of the drive particle beam is a major concern for the development of the beam-driven plasma wakefield accelerator (PWFA), and particularly for drive beams longer than the plasma wavelength ($L_b \gg \lambda_p$, where L_b is the beam length),¹⁷ such as in the proposed self-modulated proton-driven PWFA. As the long drive beam propagates in the overdense plasma ($n_b < n_0$, where n_b is the beam density) it undergoes coupled beam self-modulation (radial modulation) and beam hosing (centroid modulation). The growth rate of the hosing instability has been calculated for a long beam propagating in an overdense plasma and shown to be comparable to the growth of the self-modulational instability.¹⁸ In addition, coupling of the beam centroid motion and envelope modulations was shown to enhance the beam hosing instability in the regime of linear growth.¹⁸ Therefore, for realization of a self-modulated PWFA, it is critical to strongly seed the self-modulational instability without seeding the hosing instability.

In this work, we discuss wakefield seeding of the beam self-modulation and hosing instabilities. Controlled seeding of the instability is critical for development of a self-modulated PWFA. As well as enhancing the modulation, a controlled instability seed is required to stabilize the plasma wave amplitude and phase. Although the self-modulation and hosing growth rates are comparable (and the self-modulation instability can enhance hosing),¹⁸ we show that using an external wakefield can be effective in selectively seeding the self-modulation. In Sec. II we review the self-generated wakefield produced by a beam propagating in an overdense plasma. Section III discusses possible methods of generating a linear wakefield that will seed the instabilities. Section IV describes the evolution of the beam moments in the presence of the self-generated wakefield and the seeding wakefield. The growth of the beam self-modulation and the beam hosing in response to the seed wake is discussed in V. A summary and conclusions are presented in Sec. VI.

II. BEAM-DRIVEN WAKEFIELD

The wakefield generated by a relativistic charged particle beam moving through an overdense plasma can be calculated using linear perturbation theory of the cold plasma fluid and Maxwell equations. The evolution equation for the normalized electron density perturbation

$\delta n/n_0 = (n - n_0)/n_0$ is

$$(\partial_\zeta^2 + 1) \delta n/n_0 = (q/e)n_b/n_0, \quad (1)$$

where the q is the charge of the beam particle and $\zeta = z - ct$ is the co-moving variable. Here we are considering a highly-relativistic beam $\gamma \gg 1$, such that the quasi-static approximation may be applied, and $n_b \ll n_0$. In the following, all distances will be normalized to the plasma skin depth k_p^{-1} . The ion motion is considered negligible, which will only be valid using a heavy-ion plasma for sufficiently long drive beams.¹⁹ The beam-driven wakefields in terms of normalized potentials ($\phi = e\Phi/m_e c^2$ and $\mathbf{a} = e\mathbf{A}/m_e c^2$) driven by the plasma perturbation are

$$\nabla_\perp^2 \phi = \delta n/n_0 - (q/e)n_b/n_0, \quad (2)$$

$$\nabla_\perp^2 a_z = \beta_z - (q/e)n_b/n_0, \quad (3)$$

$$\nabla_\perp^2 \mathbf{a}_\perp = \boldsymbol{\beta}_\perp, \quad (4)$$

where $\boldsymbol{\beta}$ is the electron plasma fluid velocity and the beam is propagating in the z -direction. The (Lorentz) gauge condition may be expressed as $\nabla_\perp \cdot \mathbf{a}_\perp = \partial_\zeta(\phi - a_z)$. Defining the wake potential $\psi = \phi - a_z$, and combining Eqs. (2) and (3) yields

$$(\nabla_\perp^2 - 1) \psi = \delta n/n_0, \quad (5)$$

where we have used the first integral of the linearized axial fluid momentum equation $\beta_z = -\psi = a_z - \phi$. Combining Eqs. (1) and (5) yields the evolution equation for the wake potential driven by a beam

$$(\partial_\zeta^2 + 1) (\nabla_\perp^2 - 1) \psi = (q/e)n_b/n_0. \quad (6)$$

The force on a relativistic charged particle from the plasma wave is given by the wake potential,

$$\mathbf{F}/(eE_0) = -(q/e)\nabla\psi. \quad (7)$$

To investigate beam hosing it is useful to consider a modal decomposition of the plasma wakefield. Consider a beam distribution with azimuthal modes

$$n_b = \sum_{m=0}^{\infty} \hat{n}_{bm} \cos(m\theta), \quad (8)$$

and, similarly, the wake potential

$$\psi = -(q/e) \sum_{m=0}^{\infty} \hat{\psi}_m \cos(m\theta), \quad (9)$$

such that the evolution of the wakefield modes is

$$(\partial_\zeta^2 + 1) (\nabla_r^2 - m^2/r^2 - 1) \hat{\psi}_m = -\hat{n}_{bm}/n_0. \quad (10)$$

The Green function solution to Eq. (10) is

$$\begin{aligned} \hat{\psi}_m = \int d\zeta' \sin(\zeta - \zeta') & \left[K_m(r) \int_0^r r' dr' I_m(r') \hat{n}_{bm}(r', \zeta')/n_0 \right. \\ & \left. + I_m(r) \int_r^\infty r' dr' K_m(r') \hat{n}_{bm}(r', \zeta')/n_0 \right]. \end{aligned} \quad (11)$$

For $m = 0$, we recover the well-known axisymmetric wakefields:²⁰

$$\begin{aligned} E_z/E_0 = (q/e) \partial_\zeta \hat{\psi}_0 &= (q/e) \int d\zeta' \cos(\zeta - \zeta') \\ & \times \left[K_0(r) \int_0^r r' dr' I_0(r') n_b/n_0 + I_0(r) \int_r^\infty r' dr' K_0(r') n_b/n_0 \right], \end{aligned} \quad (12)$$

and

$$\begin{aligned} (E_r - B_\theta)/E_0 = (q/e) \partial_r \hat{\psi}_0 &= -(q/e) \int d\zeta' \sin(\zeta - \zeta') \\ & \times \left[K_1(r) \int_0^r r' dr' I_0(r') n_b/n_0 - I_1(r) \int_r^\infty r' dr' K_0(r') n_b/n_0 \right]. \end{aligned} \quad (13)$$

III. METHODS FOR WAKEFIELD SEEDING

The coherent wake initiating the instabilities is negligible for a beam with scale lengths much larger than the plasma wavelength. For example, a long ($L_b \gg 1$) Gaussian beam, with density distribution $n_b \sim \exp(-\zeta^2/2L_b^2)$, excites a wakefield such that $(E_r - B_\theta)/E_0 \sim L_b \exp(-L_b^2/2) \rightarrow 0$ for $L_b \gg 1$. Absent a coherent wake, the natural seed for the beam radial modulation (self-modulation instability) and centroid displacement (hosing instability), assuming a symmetric beam without imperfections, is from Schottky noise, i.e., the finite number of beam particles (e.g., $\sim 10^{11}$). Realistic beams will contain imperfections and asymmetries due to, e.g., beam misalignments, transport errors, and asymmetries in the beam creation and delivery, that will provide a coherent seed for the instabilities. Such seeds can dominate over that from Schottky noise ($\propto 1/\sqrt{N}$, where N is the number of beam particles).

It is desirable to control the seed for the instabilities to minimize fluctuations in the amplitude and phase of the plasma wave and to initiate beam self-modulation without

exciting beam hosing. The principal method considered in this work for seeding the self-modulational instability is to inject the long beam onto a plasma wakefield, with periodic focusing and defocusing regions, excited by a seed wakefield driver. There are many possible methods of seed wakefield excitation. For example, one can consider excitation by a short electron beam or short laser pulse, with duration near resonant for the plasma density, propagating ahead of the long beam. Or, alternatively, a wakefield may be generated by a fast rise (on the order of the plasma wavelength) in beam density at the head of the long beam. A fast rise on the head of the beam could be effectively generated by an ionization front produced by a laser (locally ionizing a neutral gas) co-propagating with the long beam. The general form of the seed transverse wakefield can be expressed as

$$(E_r - B_\theta)/E_0 = A_s f_s(r) \cos(\zeta + \varphi), \quad (14)$$

where A_s is the amplitude of the seed and $f_s(r)$ is the transverse spatial distribution of the wakefield. For example, if the seed provides a nearly linear transverse force then $f_s(r) \simeq r$. Here φ is a constant that determines the timing between the seed wake and the beam. Equation (14) assumes the seed wake is axis-symmetric.

For example, if we consider the seed wakefield excited by a short relativistic electron bunch with a flat-top radial distribution $n_s(\zeta, r) = \hat{n}_s \exp[-(\zeta - \zeta_s)^2/2\sigma_z^2] \Theta(r_s - r)$, with rms bunch length σ_z and beam radius r_s , then the transverse wakefield behind the beam, for $r < r_s$, is

$$(E_r - B_\theta)/E_0 = \frac{\hat{n}_s}{n_0} \sqrt{2\pi} \sigma_z e^{-\sigma_z^2/2} r_s K_1(r_s) I_1(r) \cos(\zeta + \varphi). \quad (15)$$

As another example, if the seed is generated by a fast rise (with scale length $\ll \lambda_p$) at the head of the long beam, then the transverse field of the seed is

$$(E_r - B_\theta)/E_0 = -(q/e) \frac{\hat{n}_b}{n_0} r_s K_1(r_s) I_1(r) [1 - \cos(\zeta)], \quad (16)$$

where the beam head is at $\zeta = 0$. The first term of the right-hand side of Eq. (16) is the adiabatic plasma focusing on a long beam propagating in overdense plasma, while the second term $\propto \cos(\zeta)$ will seed the modulation instability (and hosing, provided there is a beam tilt or a beam slice with off-axis centroid).

In the next section we will consider the influence of a seed wakefield, with general form given by Eq. (14), and the beam wakefield described in Sec. II on evolution of the moments of the beam distribution. In the numerical examples we will take $A_s = 0.046$, which,

for example, corresponds to generation of a seed wakefield using an electron beam with $n_s/n_b = 0.1$, $\sigma_z = 1$, and $r_s = 1$.

IV. EVOLUTION OF BEAM MOMENTS

In this section we consider the evolution of the first and second moments of the beam transverse distribution in response to the beam wakefield and the seed wakefield. The first moment will describe the beam centroid evolution and the second moment describes the beam envelope evolution.

A. Beam centroid evolution

The wakefield produced by the drive beam or seed can deflect the beam leading to the hosing instability. The transverse equation of motion for a single relativistic particle is $d^2\mathbf{x}/dz^2 = (m_e/\gamma M_b)(\mathbf{F}/eE_0) = \mathbf{W}_\perp$, neglecting acceleration, where $\gamma \gg 1$ is the Lorentz factor of the beam, M_b is the mass of the beam particle, and \mathbf{F} is the force owing to the wakefields. Defining the beam centroid as $x_c = \langle x \rangle$, where the brackets indicate an average over the beam transverse distribution at a beam slice ζ , yields the beam centroid evolution equation

$$\frac{d^2x_c}{dz^2} = \frac{m_e}{\gamma M_b} \langle F_x/eE_0 \rangle = -(q/e) \frac{m_e}{\gamma M_b} \langle \partial_x \psi \rangle + \langle W_{x,s} \rangle, \quad (17)$$

where $W_{x,s} = (m_e/\gamma M_b)(F_{x,s}/eE_0)$ and $F_{x,s}$ is the force owing to the seed wake in the x -direction. The wakefield of the beam can feedback to the centroid displacement, resulting in the hosing instability.¹⁸

A general (axisymmetric) transverse beam density distribution with a small [to order $\mathcal{O}(x_c/\langle x^2 \rangle^{1/2})$] centroid offset in the x -direction can be expressed as

$$n_b \left(\sqrt{[x - x_c]^2 + y^2} \right) \simeq n_{b0} - (\partial_r n_{b0}) x_c \cos \theta, \quad (18)$$

where $n_{b0} = n_b(x, y, x_c = 0) = n_b(r)$. Evaluation Eq. (17) by averaging the transverse force over the beam distribution Eq. (18) yields the equation for the centroid evolution

$$\begin{aligned} \frac{d^2x_c}{dz^2} + x_c \frac{m_e}{\gamma M_b} \frac{\int_0^\infty r dr (\partial_r n_{b0}) (\partial_r \hat{\psi}_0)}{2 \int_0^\infty r dr (n_{b0})} \\ + x_c \frac{m_e}{\gamma M_b} (q/e) A_s \cos(\zeta + \varphi) \frac{\int_0^\infty r dr (\partial_r n_{b0}) f_s(r)}{2 \int_0^\infty r dr (n_{b0})} = \frac{m_e}{\gamma M_b} \frac{\int_0^\infty dr (n_{b0}) [\partial_r (r \hat{\psi}_1)]}{2 \int_0^\infty r dr (n_{b0})} \end{aligned} \quad (19)$$

for any general axisymmetric beam distribution n_{b0} .

Assuming a flat-top transverse distribution, i.e.,

$$n_{b0} = \hat{n}_b (r_{b0}/r_b)^2 \Theta(r_b - r) f(\zeta), \quad (20)$$

with $r_{b0} = r_b(z = 0)$ and $f(\zeta)$ the normalized longitudinal profile, Eq. (19) may be evaluated to yield the centroid $x_c(\zeta, z)$ evolution equation at any beam slice (ζ):

$$\begin{aligned} \frac{d^2 x_c(\zeta)}{dz^2} = & \frac{m_e}{\gamma M_b} (q/e) A_s \cos(\zeta + \varphi) f_s(r_b) \frac{x_c}{r_b} \\ & + \frac{k_b^2}{\gamma} \frac{I_1(r_b(\zeta))}{r_b(\zeta)} \int_{-\infty}^{\zeta} d\zeta' \sin(\zeta - \zeta') f(\zeta') \frac{r_{b0}^2}{r_b(\zeta')} K_1(r_b(\zeta')) [x_c(\zeta') - x_c(\zeta)], \end{aligned} \quad (21)$$

where $k_b^2 = 4\pi\hat{n}_b e^2 / M_b c^2$ is the beam wavenumber and $r_b(\zeta, z)$ is the beam radius. In the following we will consider the long beam adiabatic regime $f \simeq 1$ (i.e., neglecting initial longitudinal beam density variations).

Assuming a linear focusing force provided by the seed $f_s(r) = r$ and the narrow beam limit, $r_b \ll 1$, the centroid evolution equation reduces to

$$\frac{d^2 x_c}{dz^2} = \frac{k_b^2}{2\gamma} \int d\zeta' \sin(\zeta - \zeta') \frac{r_{b0}^2}{r_b^2} [x_c(\zeta') - x_c(\zeta)] + \frac{m_e}{\gamma M_b} (q/e) A_s \cos(\zeta + \varphi) x_c. \quad (22)$$

B. Beam envelope evolution

The transverse wakefield will produce beam radial modulation via the periodic focusing and defocusing regions of the wakefields generated by the drive beam and the seed. These radial modulations $r_b(\zeta, z)$ couple to the beam centroid [cf. Eq. (21)]. The evolution of the rms beam transverse size (along the direction of the centroid displacement) $\sigma_x = \langle (x - x_c)^2 \rangle^{1/2}$ is given by the beam envelope equation

$$\frac{d^2 \sigma_x}{dz^2} - \frac{\epsilon_x^2}{\sigma_x^3} - \frac{1}{\sigma_x} \langle (x - x_c) W_{x,s} \rangle = -(q/e) \frac{1}{\sigma_x} \frac{m_e}{\gamma M_b} \langle (x - x_c) \partial_x \psi \rangle, \quad (23)$$

where the transverse rms geometric emittance is

$$\epsilon_x = \sqrt{\langle (x - x_c)^2 \rangle \langle (d(x - x_c)/dz)^2 \rangle - \langle (x - x_c) d(x - x_c)/dz \rangle^2}. \quad (24)$$

A similar envelope equation describes the evolution of the transverse beam size orthogonal to this displacement $\sigma_y = \langle (y - y_c)^2 \rangle^{1/2}$. Although the emittance will grow as the beam undergoes the modulation and hosing instabilities, the growth rate of these instabilities is

much faster than the betatron frequency (the characteristic frequency for the evolution of the emittance), and, therefore, we may neglect beam emittance evolution in Eq. (23) during the initial linear growth of the instabilities. This assumption provides closure to the hierarchy of beam moment equations.

Assuming a displaced beam distribution that is initially axisymmetric and the centroid displacement is small $x_c < \sigma_x$, then the beam will remain axisymmetric to order $\mathcal{O}(x_c/\sigma_x)$, such that $\langle (x - x_c)F_x \rangle = \langle yF_y \rangle + \mathcal{O}(x_c/\sigma_x)^2$. Ellipticity in the beam envelope will scale, to lowest order, as $\mathcal{O}(x_c/\sigma_x)^2$. In this limit, the envelope equation may be expressed as

$$\frac{d^2 r_b}{dz^2} - \frac{\epsilon^2}{r_b^3} = \frac{4}{r_b} \langle (x - x_c)W_{x,s} \rangle - (q/e) \frac{4}{r_b} \frac{m_e}{\gamma M_b} \langle (x - x_c) \partial_x \psi \rangle, \quad (25)$$

where the beam radius is $r_b = 2\sigma_x = 2\sigma_y$ and we have defined the effective (Lapostolle) emittance $\epsilon = 4\epsilon_x = 4\epsilon_y$. For a general distribution of the form Eq. (18), the evolution equation for the beam radius at any slice ζ is

$$\frac{d^2 r_b}{dz^2} - \frac{\epsilon^2}{r_b^3} = \frac{m_e}{\gamma M_b} \frac{2}{r_b} \frac{\int_0^\infty r dr (n_{b0} r \partial_r \hat{\psi}_0)}{\int_0^\infty r dr (n_{b0})} + \frac{m_e}{\gamma M_b} (q/e) A_s \cos(\zeta + \varphi) \frac{2}{r_b} \frac{\int_0^\infty dr r^2 f_s(r) n_{b0}}{\int_0^\infty r dr (n_{b0})}, \quad (26)$$

which is valid for any initially axis-symmetric distribution n_{b0} . Assuming a flat-top transverse beam distribution Eq. (20), Eq. (26) yields

$$\begin{aligned} \frac{d^2 r_b}{dz^2} - \frac{\epsilon^2}{r_b^3} &= \frac{m_e}{\gamma M_b} (q/e) A_s \cos(\zeta + \varphi) \frac{2}{r_b} \langle r f_s(r) \rangle \\ &\quad - \frac{k_b^2}{\gamma} \frac{4I_2(r_b)}{r_b} \int_\infty^\zeta d\zeta' \sin(\zeta - \zeta') f(\zeta') \frac{r_{b0}^2}{r_b(\zeta')} K_1(r_b(\zeta')). \end{aligned} \quad (27)$$

The beam radius evolution $r_b(\zeta, z)$ via Eq. (27) couples to the centroid evolution Eq. (21). We consider the long beam limit and neglect the longitudinal beam profile $f \simeq 1$. Assuming a linear focusing force provided by the seed $f_s(r) = r$ and the narrow beam limit, $r_b \ll 1$, the envelope evolution equation reduces to

$$\frac{d^2 r_b}{dz^2} - \frac{\epsilon^2}{r_b^3} = -\frac{k_b^2}{2\gamma} r_b \int d\zeta' \sin(\zeta - \zeta') \frac{r_{b0}^2}{r_b^2} + \frac{m_e}{\gamma M_b} (q/e) A_s \cos(\zeta + \varphi) r_b. \quad (28)$$

Note that, without modulation or a seed wakefield ($A_s = 0$), a long beam propagates according to the envelope equation

$$\frac{d^2 r_b}{dz^2} - \frac{\epsilon^2}{r_b^3} + \frac{k_b^2}{\gamma} \frac{r_{b0}^2}{r_b^2} 4I_2(r_b) K_1(r_b) = 0. \quad (29)$$

The long beam equilibrium $r_{b0} = r_{\text{eq}}$ is achieved for a radius satisfying

$$\gamma\epsilon^2 = 4k_b^2 r_{\text{eq}}^3 K_1(r_{\text{eq}}) I_2(r_{\text{eq}}). \quad (30)$$

In the narrow beam limit $r_{\text{eq}} \ll 1$, the equilibrium beam radius Eq. (30) reduces to $r_{\text{eq}}^2 = \epsilon/k_\beta$ with $k_\beta = k_b/\sqrt{2\gamma}$.

V. SEEDED BEAM SELF-MODULATION AND HOSING

In this section we examine the growth of the instability initiated by a seed wakefield. The beam centroid and envelope evolution owing to the coupling to the modulated beam-driven wakefield is assumed a perturbation to the seed-wakefield-induced evolution and the linear growth of the instabilities are calculated.

A. Seeded beam self-modulation

In the presence of the seed wakefield the beam radius will undergo betatron motion (slow compared to the instability growth) $r_0(\zeta, z)$. Using Eq. (28), the long time scale evolution of the beam radius will be driven by the seed wakefield, and the beam radius at each slice will approximately satisfy

$$\frac{d^2 r_0}{dz^2} - \frac{\epsilon^2}{r_0^3} + k_\beta^2 r_0 \simeq \frac{m_e}{\gamma M_b} (q/e) A_s \cos(\zeta + \varphi) r_0, \quad (31)$$

where we have assumed an approximately linear focusing force due to the seed wakefield $(2/r_b)\langle r f_s(r) \rangle \simeq r_b$. For simplicity we will consider a beam initially in the long-beam equilibrium (without the external seed): $r_{\text{eq}}^2 = r_{b0}^2 = \epsilon/k_\beta$. At early times, $k_\beta z \ll 1$, the beam radius at any beam slice ζ is

$$r_0(\zeta, z) \simeq r_{b0} [1 + a_s \cos(\zeta + \varphi) (k_\beta z)^2], \quad (32)$$

where $a_s = (n_0/n_b)(q/e)A_s$. As Eq. (32) shows, the beam radial modulation amplitude initially grows quadratically with respect to propagation distance in the presence of a seed wakefield (without the feedback generating the self-modulation instability). Hence, for short-beam self-modulation experiments, the observed beam modulation can be dominated by the wake from the head of the beam, i.e., the seed, and not from the self-modulation instability

growth. This is illustrated in Fig. 1, where the radial modulation amplitude growth due to the seed wakefield [curve (c)] is shown versus normalized propagation distance $k_\beta z$. The points in Fig. 1, curve (c) are from the numerical solution to Eq. (27) without the instability feedback and the solid curve (c) is Eq. (32) for the parameters: $r_{b0} = 0.2$, $n_b/n_0 = 0.008$, $\gamma = 480$ (proton beam), and seed amplitude $A_s = 0.046$. Note that, without the instability feedback, the modulation amplitude is independent of the bunch length.

Consider a perturbation r_1 from the slow betatron motion describing the beam radius self-modulation instability: $r_b = r_0 + r_1$ with $r_1 \ll r_0 < 1$. If the self-modulational instability growth length is much shorter than the betatron scale length $|\partial_z r_1| \gg k_\beta r_1$, then, using Eq. (28), the evolution equation for the perturbation in the narrow beam regime is

$$\partial_z^2 r_1 = 2k_\beta^2 \int d\zeta' \sin(\zeta - \zeta') r_1(\zeta'), \quad (33)$$

and $r_0 \simeq r_{b0}$ on the fast instability time scale. Applying the plasma wave operator and assuming a slowly varying envelope approximation, $r_1 = \hat{r} \exp(i\zeta)/2 + \text{c.c.}$ with $|\partial_\zeta \hat{r}| \ll |\hat{r}|$, yields

$$(\partial_\zeta \partial_z^2 + ik_\beta^2) \hat{r} = 0. \quad (34)$$

The initial and boundary conditions generated by the seed wakefield, Eq. (32), are $\hat{r}(z = 0, \zeta) = 0$, $\hat{r}(z, \zeta = 0) = r_{b0} a_s e^{i\varphi} (k_\beta z)^2 \Theta(z)$, and $\partial_z \hat{r}(z = 0, \zeta) = 0$. With these initial and boundary conditions, the series solution to Eq. (34) is

$$\hat{r} = r_{b0} a_s e^{i\varphi} (k_\beta z)^2 \sum_{n=0}^{\infty} \frac{2(i|\zeta|k_\beta^2 z^2)^n}{n!(2n+2)!}. \quad (35)$$

The asymptotic solution for the beam radius perturbation $r_1 = r_b - r_{b0}$ is

$$r_1 \simeq r_{b0} a_s \left(\frac{3^{13/4}}{2^4 \sqrt{2\pi}} \right) (k_\beta z)^2 \frac{e^N}{N^{5/2}} \cos \left(\frac{5\pi}{12} - \zeta - \frac{N}{\sqrt{3}} - \varphi \right), \quad (36)$$

where the number of e-foldings for the self-modulational instability is¹⁴

$$N = (3^{3/2}/4) (2|\zeta|k_\beta^2 z^2)^{1/3}. \quad (37)$$

In this model, there must be some initial seed wakefield amplitude A_s from which the instability grows. Figure 1 shows the amplitude of the beam radius modulation versus normalized propagation distance $k_\beta z$ for beam lengths (a) $L_b = 577$ and (b) $L_b = 42$, for the parameters $r_{b0} = 0.2$, $n_b/n_0 = 0.008$, $\gamma = 480$ (proton beam), and seed amplitude

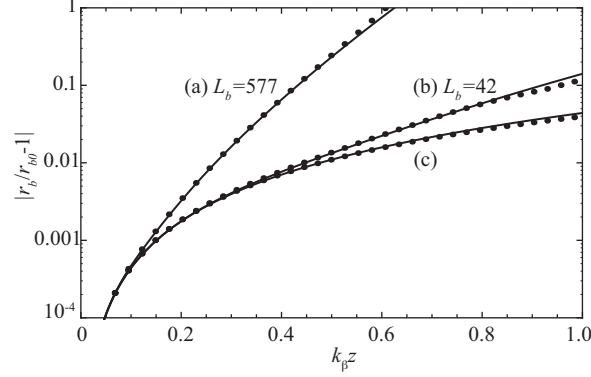


FIG. 1. Amplitude of beam radius modulation $|r_b/r_{b0} - 1|$ vs propagation distance $k_\beta z$ with beam-plasma parameters $n_b/n_0 = 0.008$, proton beam with $\gamma = 480$, $r_{b0} = 0.2$, seed wakefield amplitude $A_s = 0.046$, and bunch lengths (a) $L_b = 577$ and (b) $L_b = 42$. Points are the numerical solution to Eq. (27) and curves are the analytic solution Eq. (35). Curve (c) is modulation from seed without instability feedback Eq. (32). (Without the instability feedback, the modulation amplitude driven by the seed is independent of bunch length.)

$A_s = 0.046$. As shown in Fig. 1, for short beam lengths the modulation can be dominated by the seed wakefield for much of the propagation [cf. curves (b) and (c) of Fig. 1]. For long bunch lengths [curve (a) of Fig. 1] the modulation is dominated by the instability feedback. Here the points are numerical solutions to Eq. (27) and the solid curves (a) and (b) are the analytic solution Eq. (35).

B. Seeded beam hosing

We now consider the growth of the beam hosing instability initiated by an external seed wakefield. For definiteness consider an axis defined by the seed wakefield (e.g., driven by an short beam or laser) and the long trailing (self-modulated) beam initially displaced with respect to this axis: $x_c(z = 0, \zeta) = x_{c0}$. In practice, this small displacement may be the result of a finite alignment error. The physical effects described by this geometry are the same as a beam with a tilt, where the head with a fast rise time drives a seed wakefield and the tail is displaced from the axis of the seed wakefield driven by the beam head.

In the presence of the seed wakefield the beam centroid of the displaced long beam will undergo slow betatron motion (compared to the instability growth) $x_0(\zeta, z)$. In the following

we will assume a narrow beam $r_b < 1$. Using Eq. (22), the centroid evolution due to the seed wakefield is

$$\left[\frac{d^2}{dz^2} - \frac{m_e}{\gamma M_b} \frac{f_s(r_b)}{r_b} (q/e) A_s \cos(\zeta + \varphi) \right] x_0 = 0, \quad (38)$$

with the solution $x_0 = x_{c0} \cosh[\sqrt{2a_s \cos(\zeta + \varphi)}(k_\beta z)]$, where we have assumed an approximately linear focusing force due to the seed wakefield $f_s(r_b) \simeq r_b$. For early times, $k_\beta z \ll 1$,

$$x_0 \simeq x_{c0} [1 + a_s \cos(\zeta + \varphi)(k_\beta z)^2]. \quad (39)$$

The beam centroid at each slice is displaced toward or away from the axis depending on the position ζ with respect to the phase of the seed wakefield.

Consider a perturbation x_1 describing the beam hosing instability $x_c = x_0 + x_1$. If the hosing instability growth length is much shorter than the betatron scale length $|\partial_z x_1| \gg k_\beta |x_1|$, then the linearized evolution equation for the centroid perturbation is

$$\partial_z^2 x_1 = k_\beta^2 \int d\zeta' \sin(\zeta - \zeta') x_1(\zeta'), \quad (40)$$

and $x_0 \simeq x_{c0}$. Here we neglect the coupling between the radial modulation and centroid displacement. Applying the plasma wave operator and assuming a slowly varying envelope approximation, $x_1 = \hat{x} \exp(i\zeta)/2 + \text{c.c.}$ with $|\partial_\zeta \hat{x}| \ll |\hat{x}|$, yields

$$(\partial_\zeta \partial_z^2 + ik_\beta^2/2) \hat{x} = 0. \quad (41)$$

The initial and boundary conditions generated by the seed wakefield Eq. (39) are $\hat{x}(z = 0, \zeta) = 0$, $\hat{x}(z, \zeta = 0) = x_{c0} a_s e^{i\varphi} (k_\beta z)^2 \Theta(z)$, and $\partial_z \hat{x}(z = 0, \zeta) = 0$. With these initial and boundary conditions, the series solution to Eq. (41) is

$$\hat{x} = x_{c0} a_s e^{i\varphi} (k_\beta z)^2 \sum_{n=0}^{\infty} \frac{2(i|\zeta|k_\beta^2 z^2/2)^n}{n!(2n+2)!}. \quad (42)$$

The asymptotic solution for the beam centroid displacement $x_1 = x_c - x_{c0}$ is

$$x_1 \simeq x_{c0} a_s \left(\frac{3^{13/4}}{2^4 \sqrt{2\pi}} \right) (k_\beta z)^2 \frac{e^{N_h}}{N_h^{5/2}} \cos \left(\frac{5\pi}{12} - \zeta - \frac{N_h}{\sqrt{3}} - \varphi \right), \quad (43)$$

where the number of e-foldings for the hosing instability is¹⁸

$$N_h = (3^{3/2}/4) (|\zeta|k_\beta^2 z^2)^{1/3}. \quad (44)$$

To avoid strongly altering the structure of the self-modulated beam-driven plasma wave requires the centroid displacement to be small compared to the beam radius $x_c \ll r_{b0}$. This

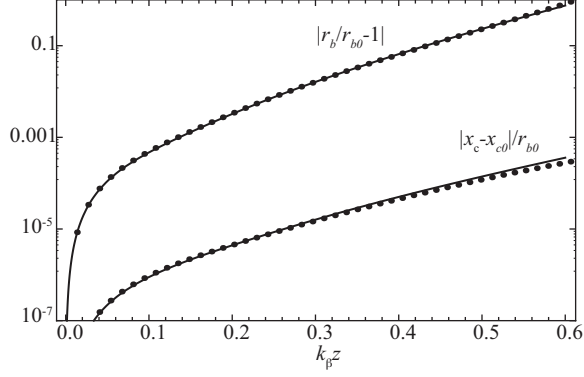


FIG. 2. Amplitude of beam radius modulation $|r_b/r_{b0} - 1|$ and beam centroid displacement $|(x_c - x_{c0})/r_{b0}|$ vs propagation distance $k_\beta z$ with beam-plasma parameters $n_b/n_0 = 0.008$ (proton beam), $\gamma = 480$, $L = 577$, $r_{b0} = 0.5$, initial displacement from the seed axis $x_{c0}/r_{b0} = 0.002$, and seed amplitude $A_s = 0.046$.

sets a tolerance for the initial displacement x_{c0} given the beam-plasma parameters and the required propagation distance.

Figure 2 shows the amplitude of the beam radius modulation $|r_b/r_0 - 1|$ and beam centroid displacement $|(x_c - x_{c0})/r_{b0}|$ vs propagation distance $k_\beta z$ with beam-plasma parameters $n_b/n_0 = 0.008$ (proton beam), $\gamma = 480$, $L_b = 577$, $r_{b0} = 0.2$, seed wakefield amplitude $A_s = 0.046$, and an initial displacement from the seed axis $x_{c0}/r_{b0} = 0.002$. The points are solutions to the coupled equations Eqs. (27) and (21), while the solid curves are the analytic solutions Eqs. (35) and (42). Figure 2 illustrates a case where the initial displacement from the seed axis is sufficiently small (compared to the initial beam radius) so that the seed wakefield enables the beam to be fully modulated $|r_b/r_{b0} - 1| \sim 1$ after $k_\beta z = 0.6$, while the centroid displacement remains small compared to the beam radius $|x_c/r_{b0}| \ll 1$ after this propagation distance.

VI. SUMMARY AND CONCLUSIONS

The self-modulation of a long beam ($L_b \gg 1$) in an overdense plasma is a method to generate a large amplitude plasma wave for particle acceleration. This regime is of interest for proton-beam-driven PWFAs, where generation of short proton beams ($\sigma_z \lesssim 100 \mu\text{m}$) is difficult. Realization of a proton-beam-driven self-modulated PWFA requires overcoming

several experimental challenges. For example, plasma inhomogeneities may suppress the instability or lead to dephasing of a witness beam through variations in the plasma wavelength, and these effects set challenging plasma uniformity constraints.¹⁶ Sufficiently heavy ions must be used to prevent ion motion limiting the wakefield generation.¹⁹ To enable controlled injection of a witness bunch, the phase of the plasma wave must be controlled by having a reproducible seed for the instability.

The phase velocity of the wakefield generated by the self-modulating beam is determined by the instability growth, and a relativistic witness bunch will dephase over a few instability e-folding lengths owing to the slow phase velocity.¹⁴ One may consider plasma tapering to control the phase velocity.¹⁶ A potentially simpler approach is to inject a witness bunch for energy gain after the beam is fully modulated, when the drive beam is no longer strongly evolving.

Associated with self-modulation is the beam hosing instability, and the hosing and self-modulation instabilities have comparable growth rates.¹⁸ Hence, the radial modulation must be strongly seeded while minimizing the seed for the hosing instability. In this work we have examined the growth of the beam self-modulation and hosing instabilities originating from a coherent seed wakefield. It is shown that an external wakefield can be effective in selectively seeding the beam radial self-modulation. This enables the beam to become fully modulated before strong beam hosing develops.

To avoid large beam centroid displacements at saturation of the radial self-modulation requires the initial centroid displacement (e.g., misalignment with respect to the seed wakefield) to be much smaller than the beam radius $x_{c0} \ll r_{b0}$. For a beam with a sharp rise at the head of the beam, this is equivalent to requiring the beam tilt (with respect to the propagation axis) to be sufficiently small $dx_c/d\zeta \ll r_{b0}/L_b$. In general, hosing will continue after saturation of the self-modulation instability, and the required propagation distance for energy transfer to a witness bunch will determine the tolerance for the initial beam misalignment to limit hosing growth over that distance.

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Appendix: Self-modulation and hosing in 2D cartesian geometry

In this Appendix we consider beam self-modulation and hosing in two-dimensional (2D) cartesian geometry. This regime is often employed when modeling non-axis-symmetric effects (e.g., beam hosing) at reduced computational expense (compared to 3D modeling). 2D cartesian geometry is also useful to study large perturbations since the beam moment equations may be derived without assuming a small beam centroid perturbation. In the following we derive the linear plasma response to the beam and the self consistent beam centroid and envelope evolution in 2D cartesian geometry.

For simplicity consider a beam in (y, z) -cartesian geometry, and in the following all length scales are normalized to the plasma skin depth k_p^{-1} . The equation for the wake potential is

$$(\partial_\zeta^2 + 1) (\partial_y^2 - 1) \psi = (q/e)(n_b/n_0), \quad (\text{A.1})$$

with the Green function solution

$$\psi = -(q/e) \int d\zeta' \sin(\zeta - \zeta') \int dy' \frac{1}{2} e^{y_{<}} e^{-y_{>}} (n_b/n_0), \quad (\text{A.2})$$

where $y_{<}$ ($y_{>}$) denotes the smaller (larger) of y and y' , respectively. We will consider the long beam regime $L_b \gg 1$ where the longitudinal variation of the beam density may be neglected. For a properly normalized transverse flat-top distribution

$$n_b = \hat{n}_b \frac{y_0}{y_e} \Theta[y_e - (y - y_c)], \quad (\text{A.3})$$

where $y_c = \langle y \rangle$ is the beam centroid position and y_e is the radius of the beam envelope, the wake potential is

$$\psi = -(q/e) \int d\zeta' \sin(\zeta - \zeta') \frac{\hat{n}_b y_0}{n_0 y_e} H_y, \quad (\text{A.4})$$

where

$$H_y = \begin{cases} e^{-(y-y_c)} \sinh(y_e), & y_c + y_e < y \\ 1 - e^{-y_e} \cosh(y - y_c), & y_c - y_e < y < y_c + y_e \\ e^{(y-y_c)} \sinh(y_e), & y < y_c - y_e. \end{cases} \quad (\text{A.5})$$

The transverse force on the beam is $F_y/(eE_0) = -(q/e)\partial_y\psi$.

The equation for the evolution of the beam centroid y_c is given by Eq. (17). Evaluating for a flat-top distribution in 2D (in the long beam regime) yields

$$\frac{d^2 y_c}{dz^2} - \langle W_{y,s} \rangle = -\frac{k_b^2}{\gamma} \frac{\sinh(y_e)}{y_e} \int d\zeta' \sin(\zeta - \zeta') \frac{y_0}{y_e(\zeta')} e^{-y_e(\zeta')} \sinh[y_c(\zeta) - y_c(\zeta')]. \quad (\text{A.6})$$

In the limit of a small centroid displacement $y_c \ll 1$,

$$\frac{d^2 y_c}{dz^2} - \langle W_{y,s} \rangle = -\frac{k_b^2}{\gamma} \frac{\sinh(y_e)}{y_e} \int d\zeta' \sin(\zeta - \zeta') \frac{y_0}{y_e(\zeta')} e^{-y_e(\zeta')} [y_c(\zeta) - y_c(\zeta')], \quad (\text{A.7})$$

and in the limit of a narrow beam $y_e \ll 1$,

$$\frac{d^2 y_c}{dz^2} - \langle W_{y,s} \rangle = -\frac{k_b^2}{\gamma} \int d\zeta' \sin(\zeta - \zeta') \frac{y_0}{y_e(\zeta')} [y_c(\zeta) - y_c(\zeta')]. \quad (\text{A.8})$$

Linearizing Eq. (A.6), assuming a small centroid displacement $y_c \ll 1$ and fixed envelope $y_e = y_0$, yields

$$(\partial_\zeta^2 + 1) (\partial_z^2 + \mu_y k_b^2 / \gamma) y_c = (\mu_y k_b^2 / \gamma) y_c \quad (\text{A.9})$$

with $\mu_y = e^{-y_0} \sinh(y_0)/y_0$. Note that $\mu_y \simeq 1 - y_0$ for $y_0 \ll 1$. Assuming $|\partial_z y_c| \gg k_b |y_c| / \gamma^{1/2}$ and $|\partial_\zeta y_c| \ll |y_c|$, the centroid displacement grows as $y_c \sim \exp(N_{2h})$ with

$$N_{2h} = (3^{3/2}/4) (\mu_y k_b^2 \zeta^2 / \gamma)^{1/3}. \quad (\text{A.10})$$

The envelope equation is given by Eq. (23). For a transverse flat-top distribution Eq. (A.3), $\sigma_y = y_e / \sqrt{3}$, and the envelope equation is

$$\frac{d^2 y_e}{dz^2} - \frac{9\epsilon_y^2}{y_e^3} = \frac{3}{y_e} \langle (y - y_c) W_{y,s} \rangle - (q/e) \frac{3}{y_e} \frac{m_e}{\gamma M_b} \langle (y - y_c) \partial_y \psi \rangle. \quad (\text{A.11})$$

Evaluating the 2D wakefield Eq. (A.4) (in the long beam regime) yields the beam envelope equation

$$\begin{aligned} \frac{d^2 y_e}{dz^2} - \frac{9\epsilon_y^2}{y_e^3} - \frac{3}{y_e} \langle (y - y_c) W_{y,s} \rangle \\ = -\frac{k_b^2}{\gamma} \frac{3}{y_e} \left[\cosh(y_e) - \frac{\sinh(y_e)}{y_e} \right] \int d\zeta' \sin(\zeta - \zeta') \frac{y_0}{y_e(\zeta')} e^{-y_e(\zeta')} \cosh[y_c - y_c(\zeta')]. \end{aligned} \quad (\text{A.12})$$

The long beam adiabatic equilibrium is given by

$$3\epsilon_y^2 = e^{-y_0} [\cosh(y_0) - \sinh(y_0)/y_0] k_b^2 y_0^2 / \gamma, \quad (\text{A.13})$$

and for $y_0 \ll 1$, the equilibrium is $9\gamma\epsilon_y^2 = k_b^2 y_0^4$.

In the limit without displacement $y_c = 0$, Eq. (A.12) reduces to

$$\begin{aligned} \frac{d^2 y_e}{dz^2} - \frac{9\epsilon_y^2}{y_e^3} - \frac{3}{y_e} \langle (y - y_c) W_{y,s} \rangle \\ = -\frac{k_b^2}{\gamma} \frac{3}{y_e} [\cosh(y_e) - \sinh(y_e)/y_e] \int d\zeta' \sin(\zeta - \zeta') \frac{y_0}{y_e(\zeta')} e^{-y_e(\zeta')}. \end{aligned} \quad (\text{A.14})$$

And, in the limit without a centroid displacement and for a narrow beam $y_e \ll 1$,

$$\frac{d^2 y_e}{dz^2} - \frac{9\epsilon_y^2}{y_e^3} - \frac{3}{y_e} \langle (y - y_c) W_{y,s} \rangle = -\frac{k_b^2}{\gamma} y_e \int d\zeta' \sin(\zeta - \zeta') \frac{y_0}{y_e(\zeta')}. \quad (\text{A.15})$$

If we assume no centroid displacement $y_c = 0$ and perturbing Eq. (A.14) about the equilibrium $y_e = y_0 + y_1$ with $y_1 \ll y_0$, the linear growth of the beam self-modulation can be determined. In the strongly-coupled regime such that $|\partial_z y_1| \gg k_b |y_1|/\gamma^{1/2}$, the linearized envelope equation is

$$(\partial_\zeta^2 + 1) \partial_z^2 y_1 = (\nu_y k_b^2 / \gamma) y_1 \quad (\text{A.16})$$

with

$$\nu_y = 3y_0^{-2} (1 + y_0) e^{-y_0} [\cosh(y_0) - \sinh(y_0)/y_0]. \quad (\text{A.17})$$

Note that $\nu_y \simeq 1 - 2y_0^2/5$ for $y_0 \ll 1$. Assuming $|\partial_z y_1| \gg k_b |y_1|/\gamma^{1/2}$ and $|\partial_\zeta y_1| \ll |y_1|$, the beam radius perturbation grows as $y_1 \sim \exp(N_2)$ with

$$N_2 = (3^{3/2}/4) (\nu_y k_b^2 \zeta^2 / \gamma)^{1/3}. \quad (\text{A.18})$$

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